Magnetically Coupled Circuits

When two loops of a circuit with or without any physical contact affect each other with the help of magnetic fields, the circuits are said to be **magnetically coupled**. Magnetically coupled circuits are important parameters in electrical engineering. The most common example of a magnetically coupled circuit is a **transformer**. The **transformer** is a device used to transfer electric energy from one circuit to another. The primary winding and secondary winding are the main elements of a transformer. The primary winding creates varying magnetic flux and the secondary winding induces the electromotive force due to the effect of magnetic flux created by the primary winding.

Faraday's Law of Electromagnetic Induction

Faraday's law of induction (or simply Faraday's law) is a basic law of electromagnetism predicting how a magnetic field will interact with an electric circuit to produce an electromotive force (emf)—a phenomenon known as **electromagnetic induction**. It is the fundamental operating principle of transformers, inductors, and many types of electric motors, generators and solenoids.

Faraday's law states that, if a conductor is moved through a magnetic field so that it cuts magnetic lines of flux, a voltage will be induced across the conductor.

Magnetic flux is a measurement of the total magnetic field which passes through a given area. It is a useful tool for helping describe the effects of the magnetic force on something occupying a given area.

The greater the number of flux lines cut per unit time (by increasing the speed with which the conductor passes through the field), or the stronger the magnetic field strength (for the same traversing speed), the greater will be the induced voltage across the conductor. If the conductor is held fixed and the magnetic field is moved so that its flux lines cut the conductor, the same effect will be produced.

If a coil of N turns is placed in the region of a changing flux, a voltage will be induced across the coil as determined by Faraday's law:

$$e = N\frac{d\phi}{dt} \text{ volts, V}$$

Where e is voltage induced

- N represents the number of turns of the coil and
- $d\phi/dt$ is the instantaneous change in flux (in webers) linking the coil.

➢ The term linking refers to the flux within the turns of wire.

If the flux linking the coil ceases to change, such as when the coil simply sits still in a magnetic field of fixed strength, $d\phi/dt = 0$, and the induced voltage $e = N\,(d\phi/dt) = N(0) = 0$.

Self-Inductance

Self-inductance is the tendency of a coil to resist changes in current in itself. Whenever current changes through a coil, it induces an EMF, which is proportional to the rate of change of current through the coil.

Self-inductance of a coil is a measure of the change in flux linking a coil due to a change in current through the coil; that is

$$L = N\frac{d\phi}{di}$$

measured in henrys (H)

The opposition in the form of an induced voltage across the inductor is directly proportional to the time rate of change of the current. The induced voltage is given by the formula

$$e_L = v = N\frac{d\phi}{dt} = (N\frac{d\phi}{di})(\frac{di}{dt})$$

$$e_L = L\frac{di}{dt}$$

where L is the constant of proportionality called the inductance of the inductor.

The inductance of an inductor depends on its physical dimension and construction. Inductors (coils) of different shapes have different formulas: For example,

- Inductance of a solenoid - $L = N^2 \cdot \mu_0 A / l$

Mutual Inductance

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as mutual inductance. The change in current in a certain coil can also induce voltage across the terminals of another coil placed in its vicinity.

Let us first consider a single inductor, a coil with N turns. When current *i* flows through the coil, a magnetic flux φ is produced around it. According to Faraday's law, the voltage *v* induced in the coil is proportional to the number of turns *N* and the time rate of change of the magnetic flux φ; that is,

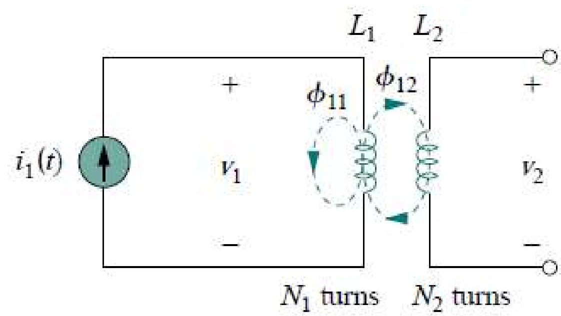
$$v = N\frac{d\phi}{dt}$$

But the flux φ is produced by current i so that any change in φ is caused by a change in the current. Thus,

$$v = N\frac{d\phi}{di}\frac{di}{dt} = L\frac{d\phi}{dt}$$

- L is inductance or commonly called self-inductance, because it relates the voltage induced in a coil by a time-varying current in the same coil.

Now consider two coils with self-inductances L_1 and L_2 that are in close proximity with each other. Coil 1 has N_1 turns, while coil 2 has N_2 turns. For the sake of simplicity, assume that the second inductor carries no current. The magnetic flux φ_1 coming from coil 1 has two components: one component φ_{11} links only coil 1, and another component φ_{12} links both coils.



Hence,

$$\varphi_1 = \varphi_{11} + \varphi_{12}$$

Since the entire flux $\varphi 1$ links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1\frac{d\phi_1}{dt}$$

Only flux φ_{12} links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2\frac{d\phi_{12}}{dt}$$

Rearranging the above equation, we have

$$v_2 = \left(N_2\frac{d\phi_{12}}{di_1}\right)\left(\frac{di_1}{dt}\right) = M_{21}\left(\frac{di_1}{dt}\right)$$

- Where $M_{21} = N_2\frac{d\phi_{12}}{di_1}$

M_{21} is known as the mutual inductance of coil 2 with respect to coil 1. Subscript 21 indicates that the inductance M_{21} relates the voltage induced in coil 2 to the current in coil 1.

Similarly, suppose we now let current i_2 flow in coil 2, while coil 1 carries no current. The magnetic flux φ_2 emanating from coil 2 comprises flux φ_{22} that links only coil 2 and flux φ_{21} that links both coils.

$$v_1 = \left(N_1 \frac{d\phi_{21}}{di_2}\right)\left(\frac{di_2}{dt}\right) = M_{12}\left(\frac{di_2}{dt}\right)$$

- Where $M_{12} = N_1 \frac{d\phi_{21}}{di_2}$

  which is the *mutual inductance* of coil 1 with respect to coil 2

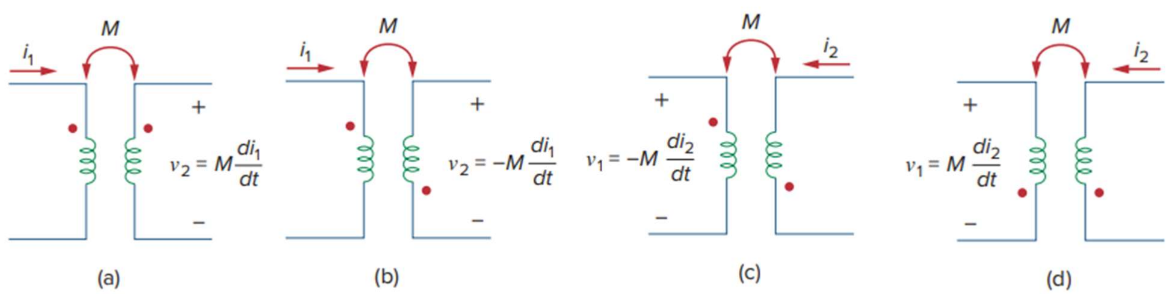
If all of the flux linking the primary links the secondary, then $\phi_1 = \phi_2$

$$v_2 = N_1 \left(\frac{d\phi_1}{dt}\right)$$

The mutual inductance between the two coils is determined by

$$\mathrm{M} = N_1 \frac{d\phi_2}{di_2} \qquad M = N_2 \frac{d\phi_1}{di_1}$$

Although mutual inductance M is always a positive quantity, the mutual voltage M di⁄dt may be negative or positive, just like the self-induced voltage L di⁄dt. However, unlike the self-induced L di⁄dt, whose polarity is determined by the reference direction of the current and the reference polarity of the voltage (according to the passive sign convention), the polarity of mutual voltage M di⁄dt is not easy to determine, because four terminals are involved. The choice of the correct polarity for M di⁄dt is made by examining the orientation or particular way in which both coils are physically wound and applying Lenz's law in conjunction with the right-hand rule. Since it is inconvenient to show the construction details of coils on a circuit schematic, we apply the dot convention in circuit analysis. By this convention, a dot is placed in the circuit at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal of the coil. This is illustrated in the figure below.



Given a circuit, the dots are already placed beside the coils so that we need not bother about how to place them. The dots are used along with the dot convention to determine the polarity of the mutual voltage. The dot convention is stated as follows:

- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.
- Alternatively, if a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

Thus, the reference polarity of the mutual voltage depends on the reference direction of the inducing current and the dots on the coupled coils. Application of the dot convention is illustrated in the four pairs of mutually coupled coils given above. For the coupled coils in (a), the sign of the mutual voltage v_2 is determined by the reference polarity for v_2 and the direction of i_1. Since i_1 enters the dotted terminal of coil 1 and v_2 is positive at the dotted terminal of coil 2, the mutual voltage is $+M\, di1/dt$. For the coils in (b), the current i_1 enters the dotted terminal of coil 1 and v_2 is negative at the dotted terminal of coil 2. Hence, the mutual voltage is $-M\, di1/dt$. The same reasoning applies to the coils in (c) and (d).

For the coils in (a), the total inductance is

$$L = L_1 + L_2 + 2M \qquad \text{(Series-aiding connection)}$$

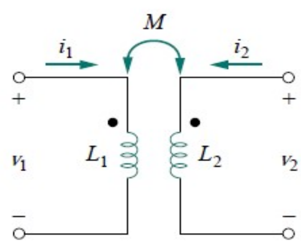
For the coils in (b), the total inductance is

$$L = L_1 + L_2 - 2M \qquad \text{(Series-opposing connection)}$$

Energy in a Coupled Circuit

The energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2$$



Consider the circuit above. We assume that currents i_1 and i_2 are zero initially, so that the energy stored in the coils is zero. If we let i_1 increase from zero to I_1 while maintaining $i_2 = 0$, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1\, di_1/\, dt$$

and the energy stored in the circuit is

$$v_1 = \int p_1\, dt = L_1 \int_0^{I_1} i_1\, di_1 = \frac{1}{2} L_1 I_1^2$$

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the mutual voltage induced in coil 1 is $\boldsymbol{M_{12}\, di_2/dt}$, while the mutual voltage induced in coil 2 is zero, since i_1 does not change. The power in the coils is now

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = I_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

and the energy stored in the circuit is

$$w_2 = \int p_2\, dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2\, di_2$$
$$= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

The total energy stored in the coils when both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1, the total energy stored in the coils is

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$$

This equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, the mutual voltage is negative, so that the mutual energy MI_1I_2 is also negative. In that case,

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2$$

The energy stored in the circuit cannot be negative because the circuit is passive. Thus, the mutual inductance cannot be greater than the geometric mean of the self-inductances of the coils.

$$M \leq \sqrt{L_1 L_2}$$

The extent to which the mutual inductance M approaches the upper limit is specified by the *coefficient of coupling k*, given by

$$k = \frac{M}{\sqrt{L_1 L_2}}$$